1 Limits and Their Properties

1.5 Infinite Limits

Objectives

- Determine infinite limits from the left and from the right.
- Find and sketch the vertical asymptotes of the graph of a function.

Infinite Limits

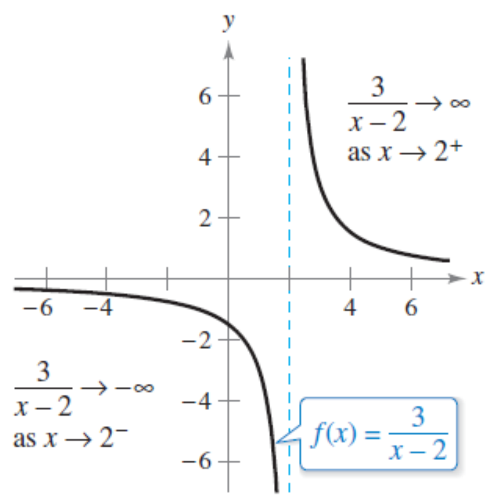
Infinite Limits

Consider the function $f(x) = 3/(x - 2)$. From Figure 1.38 and the table, you can see that $f(x)$ *decreases without bound* as x approaches 2 from the left, and $f(x)$ *increases without bound* as x approaches 2 from the right.

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	-6	-30	-300	-3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.



$f(x)$ increases and decreases without bound as x approaches 2.

Figure 1.38

Infinite Limits

This behavior is denoted as

$$\lim_{x \to 2^-} \frac{3}{x - 2} = -\infty$$

$f(x)$ decreases without bound as x approaches 2 from the left.

and

$$\lim_{x \to 2^+} \frac{3}{x - 2} = \infty.$$

$f(x)$ increases without bound as x approaches 2 from the right.

The symbols ∞ and $-\infty$ refer to positive infinite and negative infinity, respectively.

These symbols do not represent real numbers. They are convenient symbols used to describe unbounded conditions more concisely.

Infinite Limits

A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an **infinite limit.**

Definition of Infinite Limits

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x \to c} f(x) = \infty$$

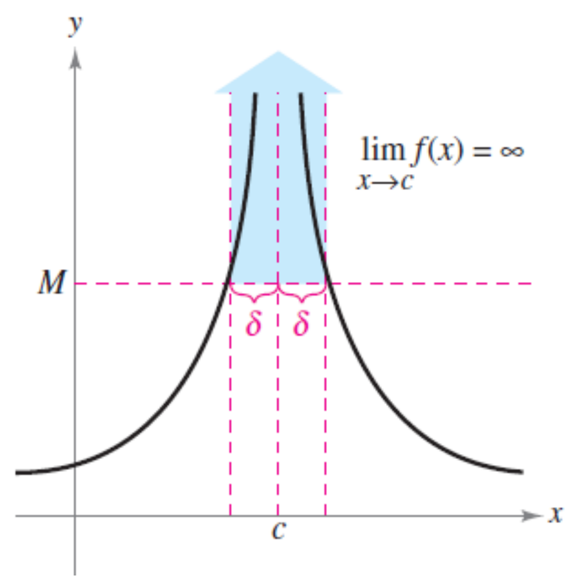
means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.39). Similarly, the statement

$$\lim_{x \to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever

$$0 < |x - c| < \delta.$$

To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.

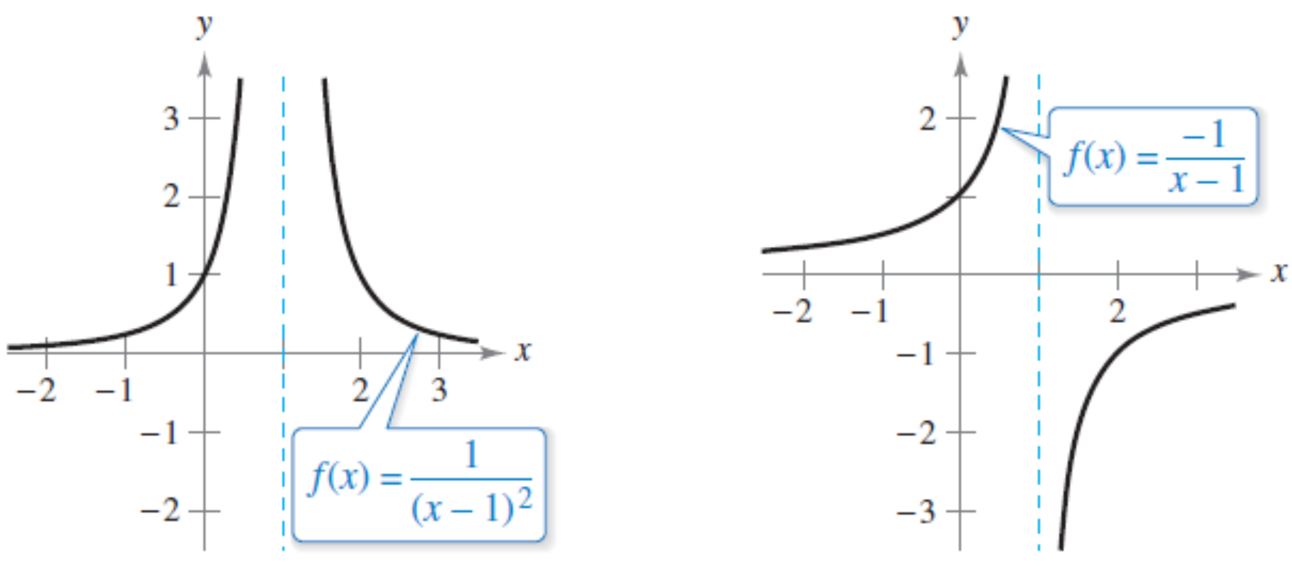


Infinite limits

Figure 1.39

Example 1 – *Determining Infinite Limits from a Graph*

Determine the limit of each function shown in Figure 1.40 as *x* approaches 1 from the left and from the right.



(a)

(b)

Each graph has an asymptote at $x = 1$.

Figure 1.40

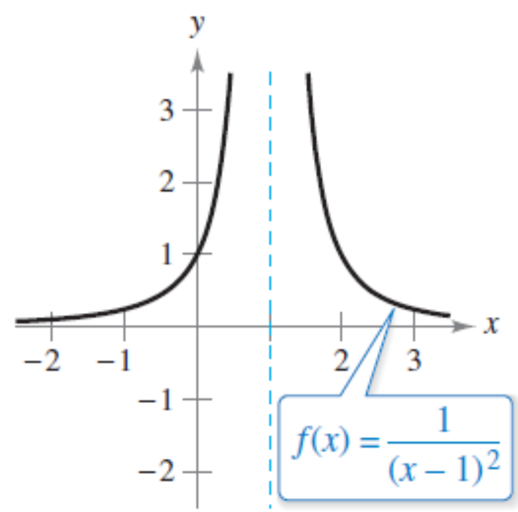
Example 1(a) – *Solution*

When x approaches 1 from the left or the right, $(x - 1)^2$ is a small positive number.

Thus, the quotient $1/(x - 1)^2$ is a large positive number, and $f(x)$ approaches infinity from each side of $x = 1$.
So, you can conclude that

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty.$$ Limit from each side is infinity.

Figure 1.40(a) confirms this analysis.



The graph has an asymptote at $x = 1$.

Figure 1.40(a)

Example 1(b) – *Solution*

cont'd

When x approaches 1 from the left, $x - 1$ is a small negative number.

Thus, the quotient $-1/(x - 1)$ is a large positive number, and $f(x)$ approaches infinity from left of $x = 1$.

So, you can conclude that

$$\lim_{x \to 1^-} \frac{-1}{x-1} = \infty.$$

Limit from the left side is infinity.

When x approaches 1 from the right, $x - 1$ is a small positive number.

Example 1(b) – *Solution*

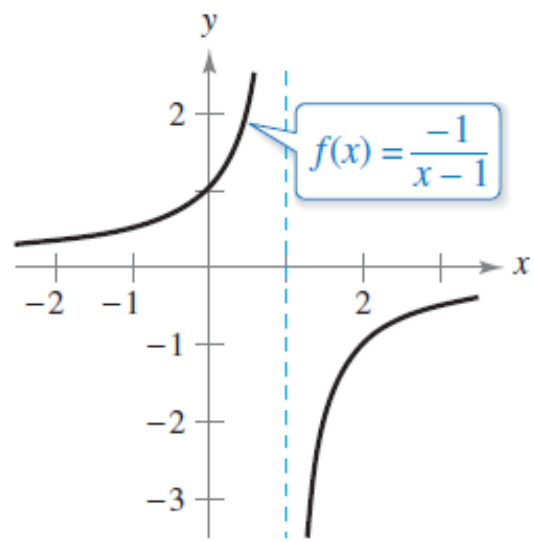
cont'd

Thus, the quotient $-1/(x - 1)$ is a large negative number, and $f(x)$ approaches negative infinity from the right of $x = 1$.

So, you can conclude that

$$\lim_{x \to 1^+} \frac{-1}{x - 1} = -\infty.$$ Limit from the right side is negative infinity.

Figure 1.40(b) confirms this analysis.



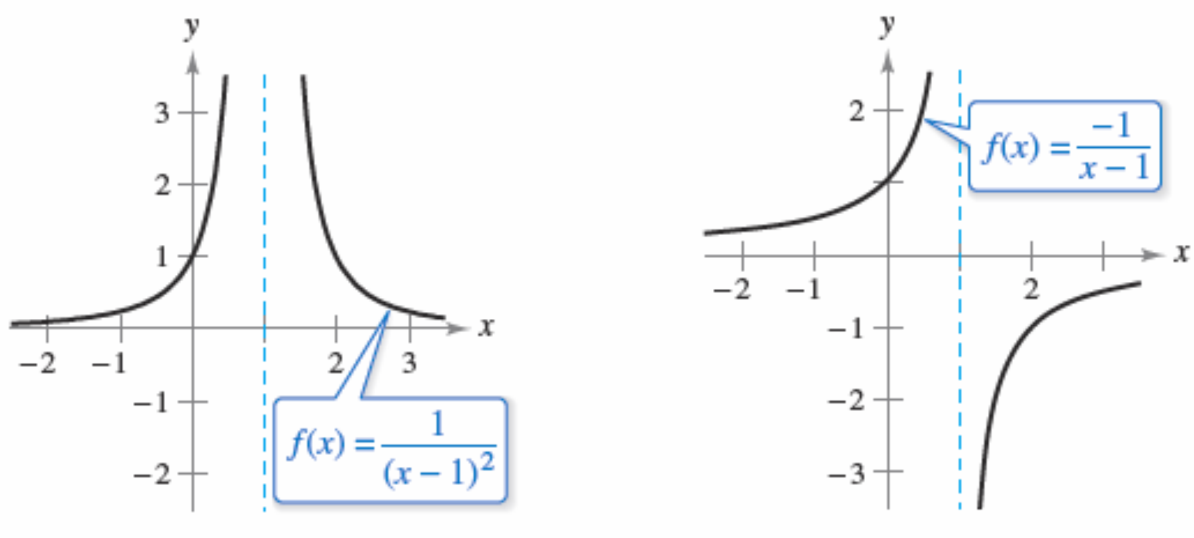
The graph has an asymptote at $x = 1$.

Figure 1.40(b)

Vertical Asymptotes

Vertical Asymptotes

If it were possible to extend the graphs in Figure 1.40 toward positive and negative infinity, you would see that each graph becomes arbitrarily close to the vertical line $x = 1$. This line is a **vertical asymptote** of the graph of f.



(a) (b)

Each graph has an asymptote at $x = 1$.

Figure 1.40

Vertical Asymptotes

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

In Example 1, note that each of the functions is a *quotient* and that the vertical asymptote occurs at a number at which the denominator is 0 (and the numerator is not 0). The next theorem generalizes this observation.

Vertical Asymptotes

THEOREM 1.14 Vertical Asymptotes

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

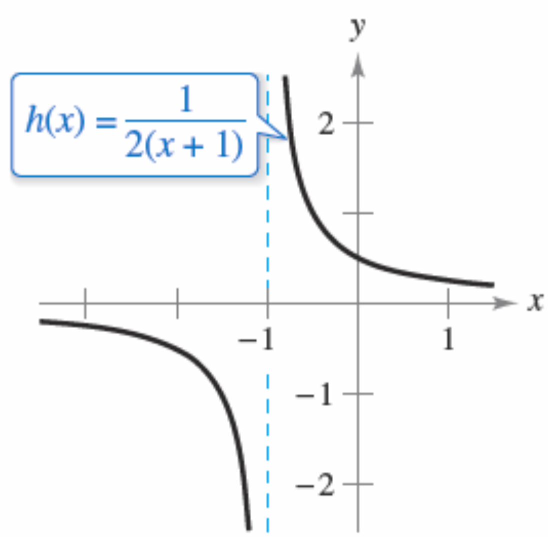
$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

Example 2 – *Finding Vertical Asymptotes*

a. When $x = -1$, the denominator of $h(x) = \frac{1}{2(x+1)}$ is 0 and the numerator is not 0.

So, by Theorem 1.14, you can conclude that $x = -1$ is a vertical asymptote, as shown in Figure 1.41(a).



Function with vertical asymptotes

Figure 1.41(a)

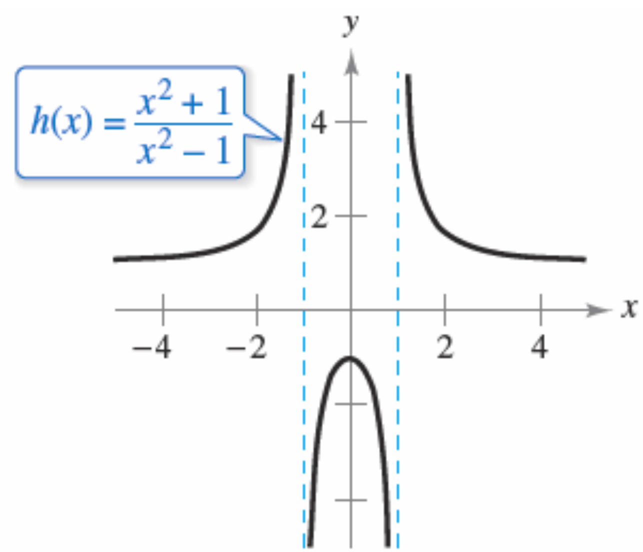
Example 2 – *Finding Vertical Asymptotes* cont'd

b. By factoring the denominator as

$$h(x) = \frac{x^2 + 1}{x^2 - 1} = \frac{x^2 + 1}{(x - 1)(x + 1)}$$

we can see that the denominator is 0 at $x = -1$ and $x = 1$.

Also, because the numerator is not 0 at these two points, you can apply Theorem 1.14 to conclude that the graph of *f* has two vertical asymptotes, as shown in figure 1.41(b).



Function with vertical asymptotes

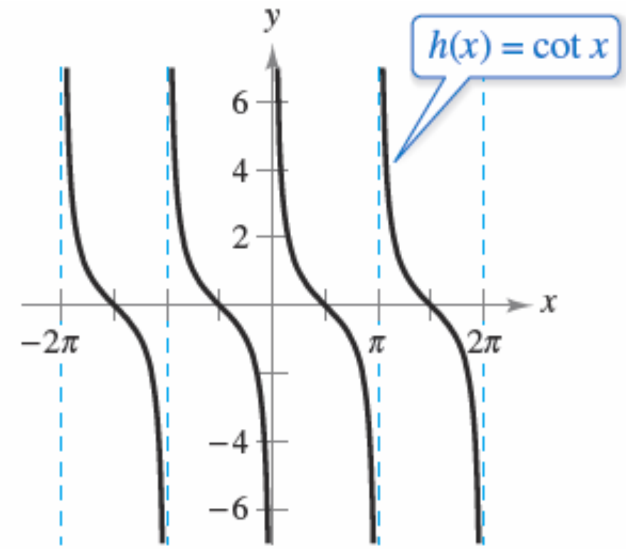
Figure 1.41(b)

Example 2 – *Finding Vertical Asymptotes* cont'd

c. By writing the cotangent function in the form

$$h(x) = \cot x = \frac{\cos x}{\sin x}$$

we can apply Theorem 1.14 to conclude that vertical asymptotes occur at all values of *x* such that sin *x* = 0 and cos *x* ≠ 0, as shown in Figure 1.41(c).



Function with vertical asymptotes

Figure 1.41(c)

So, the graph of this function has infinitely many vertical asymptotes. These asymptotes occur at $x = n\pi$, where n is an integer.

Vertical Asymptotes

THEOREM 1.15 Properties of Infinite Limits

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L.$$

1. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = \infty$

2. Product: $\lim_{x \to c} [f(x)g(x)] = \infty, \quad L > 0$

$\lim_{x \to c} [f(x)g(x)] = -\infty, \quad L < 0$

3. Quotient: $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Example 5 – *Determining Limits*

a. Because $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0} \frac{1}{x^2} = \infty$, you can write

$$\lim_{x\to 0}\left(1 + \frac{1}{x^2}\right) = \infty. \qquad \text{Property 1, Theorem 1.15}$$

b. Because $\lim_{x\to 1^-} (x^2 + 1) = 2$ and $\lim_{x\to 1^-} (\cot \pi x) = -\infty$, you can write

$$\lim_{x\to 1^-} \frac{x^2 + 1}{\cot \pi x} = 0. \qquad \text{Property 3, Theorem 1.15}$$

c. Because $\lim_{x\to 0^+} 3 = 3$ and $\lim_{x\to 0^+} \cot x = \infty$, you can write

$$\lim_{x\to 0^+} 3 \cot x = \infty. \qquad \text{Property 2, Theorem 1.15}$$

d. Because $\lim_{x\to 0^-} x^2 = 0$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, you can write

$$\lim_{x\to 0^-}\left(x^2 + \frac{1}{x}\right) = -\infty. \qquad \text{Property 1, Theorem 1.15}$$